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HEEGNER POINTS, CYCLES AND MAASS FORMS

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ABSTRACT

We derive for Hecke–Maass cusp forms on the full modular group a relation between the sum of the form at Heegner points (and integrals over Heegner cycles) and the product of two Fourier coefficients of a corresponding form of half-integral weight. Specializing to certain cycles we obtain the non-negativity of the L-function of such a form at the center of the critical strip. These results generalize similar formulae known for holomorphic forms.

0. Introduction

In a somewhat neglected paper [M] Maass discovered that the sum of a Maass cusp form φ over the "Heegner points" of a given discriminant d is equal to the dth Fourier coefficient of a Maass form of $\frac{1}{2}$-integral weight. See also more recent work of Hejhal [H]. Later after Shimura's fundamental paper [S] on $\frac{1}{2}$-integral weight forms, Shintani [Sh], Niwa [N], Waldspurger [W], Kohnen [Ko1], [Ko2]

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and Kohnen and Zagier [KoZ] studied this phenomenon for holomorphic forms. In particular, it was found that the constant of proportionality (after suitable normalizations) in the above relation is also a Fourier coefficient! Our purpose in this paper is to return to the Maass case and develop a precise form of the identity in the non-holomorphic case. Our result has a number of applications. Firstly to the equidistribution of the Heegner points. Following the analysis in Duke [D] one can use the precise form to develop sharp forms of equidistribution. Secondly we prove that the L-function of a Maass–Hecke cusp form for $\Gamma = \mathrm{SL}_2(\mathbb{Z})$ is non-negative at the center of its critical strip. Note that this non-trivial fact follows immediately from the Riemann Hypothesis for the L-function. The analogous result in the holomorphic case is due to Waldspurger [W] and Kohnen–Zagier [KoZ].

The new identity may be viewed as an extension to Maass cusp forms of a well-known relation for Eisenstein series.

Let $\Gamma = \mathrm{SL}_2(\mathbb{Z})$ denote the modular group. It acts on integral binary quadratic forms $ax^2 + bxy + cy^2$ in the usual way and leaves the discriminant $d = b^2 - 4ac$ invariant. For a fixed $d \equiv 0, 1 \pmod 4$ let Λ_d denote a complete set of Γ inequivalent forms of discriminant d. If $d < 0$ we associate to each $ax^2+bxy+cy^2$ the point $z \in \mathfrak{H} = \{z \mid \operatorname{Im}(z) > 0\}$, $z = \frac{-b\pm\sqrt{d}}{2a}$. In this way we get $h(d) = |\Lambda_d|$ points in $\Gamma\backslash\mathfrak{H}$ which we refer to as the Heegner points of discriminant d (these will also be denoted by Λ_d). We shall denote the sum over Heegner points weighted by one over the order of the stabilizer by $\sum^*_{z\in\Lambda_d}$. Similarly if $d > 0$ we obtain geodesic cycles in $\Gamma\backslash\mathfrak{H}$ which we refer to as Heegner cycles.

The Eisenstein series for Γ is defined by

$$E(z,s) = \sum_{m,n}{}' \frac{v^s}{|mz+n|^{2s}}, \quad z = u + iv \in \mathfrak{H}. \tag{0.1}$$

Let χ_d be the Dirichlet character $\chi_d(n) = (\frac{d}{n})$ where $(\frac{d}{n})$ is the Kronecker symbol [S]. For d a fundamental discriminant (say $d < 0$) the following is well known:

$$\sum_{z\in\Lambda_d}{}^{*} E(z,s) = \zeta(s)L(s,\chi_d) \tag{0.2}$$

where

$$\zeta(s) = \sum_{n=1}^{\infty} n^{-s}, \quad L(s,\chi_d) = \sum_{n=1}^{\infty} \frac{\chi_d(n)}{n^s}. \tag{0.3}$$

Now the right hand side in (0.2) may be interpreted as a product of the first and the dth Fourier coefficients of the Eisenstein series of $\frac{1}{2}$-integral weight [GH]. From this point of view our identity generalizes (0.2) with $E(z,s)$ being replaced by a Maass cusp form.

Let

$$(0.4)\qquad \begin{aligned} U =& L^2_{\mathrm{cusp}}(\Gamma\backslash\mathfrak{H}) = \{f\colon \mathfrak{H}\to\mathbb{C} \mid f(\gamma z)=f(z) \quad \text{for } \gamma\in\Gamma, \\ & \int_{\Gamma\backslash\mathcal{H}} |f|^2\frac{dxdy}{y^2}<\infty, \int_0^1 f(x,y)dx = 0 \ \text{ for a.e. } y\}. \end{aligned}$$

Then U is a Hilbert space with the obvious inner product $\langle\ ,\ \rangle$; it is invariant by the actions of the Laplace operator $\Delta_0 = y^2(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2})$ and by the Hecke operators T_m, $m=2,3,\ldots$ (see §1). These operators commute and there is an orthogonal basis of U consisting of simultaneous eigenforms for Δ_0 and T_m. Such eigenforms will be called **Maass–Hecke forms**.

Similarly we introduce a Hilbert space

$$(0.5)\qquad \begin{aligned} V =& L^2_{\mathrm{cusp}}(\Gamma_0(4)\backslash\mathfrak{H}, J) \\ =& \{f\colon \mathfrak{H}\to\mathbb{C} \mid f(\gamma z) = J(\gamma,z)f(z) \text{ for } \gamma\in\Gamma_0(4), \\ & f \ \text{ cuspidal and square integrable}\}. \end{aligned}$$

Here $\Gamma_0(4) = \{\gamma\in\Gamma \mid \gamma = \begin{bmatrix} a & b \\ c & d\end{bmatrix},\ 4|c\}$, $J(\gamma,z) = \frac{\theta(\gamma z)}{\theta(z)}$, $\theta(z) = y^{1/4}\theta_1(z) = y^{1/4}\sum_{n=-\infty}^{\infty} e(n^2 z)$ with the usual abbreviation $e(z)=e^{2\pi i z}$, and cuspidal means the zeroth Fourier coefficient is 0 in each of the 3 cusps of $\Gamma_0(4)\backslash\mathfrak{H}$. This time the Laplacian $\Delta_{1/2}$ and Hecke operators T_{p^2}, $p\neq 2$ leave V invariant, are self-adjoint and commute with each other. Here

$$(0.6)\qquad \Delta_{1/2} = y^2\Big(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\Big) - \frac{1}{2}iy\frac{\partial}{\partial x}.$$

The linear operators τ_2 and σ map $V\to V$ where

$$(0.7)\qquad \tau_2 F(z) = e^{i\pi/4}\Big(\frac{z}{|z|}\Big)^{-1/2} F(-1/4z),$$

$$(0.8)\qquad \sigma F(z) = \frac{\sqrt{2}}{4}\sum_{\nu \bmod 4} F\Big(\frac{z+\nu}{4}\Big).$$

If $L=\tau_2\sigma$ then $L\colon V\to V$ is in fact self-adjoint and commutes with $\Delta_{1/2}$ and T_{p^2}. Thus we can find an orthonormal basis of V of simultaneous eigenforms

$F_1, F_2, \ldots$ of $L, \Delta_{1/2}, T_{p^2}$, $p \neq 2$. The subspace V^+ of V on which $LF = F$ will play a central and similar role to Kohnen's space $S^{(1)}_{k+1/2}$ [Ko1]. The F_j's are Maass forms of weight $\frac{1}{2}$.

Maass forms in U or V are invariant by $z \to z+1$ and so have Fourier developments. From the differential equation

$$\Delta_k f + \lambda f = 0, \quad k = 0 \text{ or } \frac{1}{2} \quad (\text{the "weight"}) \tag{0.9}$$

and the fact that f is square integrable one finds a development of the form

$$f(z) = \sum_{n \neq 0} b(n) W_{\frac{k}{2}\mathrm{sgn}(n), ir}(4\pi |n| y) e(nx) \tag{0.10}$$

where $\frac{1}{4} + r^2 = \lambda$ and W is the usual Whittaker function [MO] which is normalized so that

$$W_{\beta,\mu}(y) \sim e^{-y/2} y^{\beta} \quad \text{as } y \to \infty. \tag{0.11}$$

The numbers $b(n)$ above are uniquely determined and called **the Fourier coefficients of** f. We will denote the Fourier coefficients of the weight $\frac{1}{2}$ eigenbasis F_j by $\rho_j(n)$.

Concerning a non-zero Maass–Hecke eigenform φ of weight 0 (i.e. in U) we can (by Proposition 1.2) and will always normalize φ so that its first coefficient $b(1) = 1$—we call this **Hecke normalized.** The L-**function** associated with a Maass form $\varphi \in U$ is defined by

$$L(\varphi, s) = \sum_{n=1}^{\infty} \frac{b(n)}{n^{s-1/2}}, \tag{0.13}$$

where $b(n)$ are Fourier coefficients of φ.

Given one of the F_j's$\in V$ we define

$$\psi(w) = \int_{\Gamma_0(4)\backslash \mathfrak{H}} F_j(z) \overline{\Theta(z, w)} \frac{du dv}{v^2}$$

where $\Theta(z, w)$ is the Siegel's Θ-function discussed in §2. Then by Proposition 4.1, $\psi \in U$. If $\rho_j(1) = 0$, $\psi = 0$; while if $\rho_j(1) \neq 0$, it is a Maass–Hecke eigenform and hence can be Hecke normalized. Such a Hecke normalized form $\varphi \in U$ will be called the **Shimura lift** of F_j and will be denoted by Shim (F_j):

$$\varphi \triangleq \mathrm{Shim}(F_j) = \sum_{n \neq 0} a(n) W_{0, 2ir_j}(4\pi |n| y) e(nx) \tag{0.14}$$

where $\lambda_j = \frac{1}{4} + r_j^2$ (for F_j). It is an even form (i.e. $a(-n) = a(n)$) and its Fourier coefficients satisfy

$$\zeta(s+1)\sum_{n=1}^{\infty}\frac{\rho_j(n^2)}{n^{s-1/2}} = \rho_j(1)\sum_{n=1}^{\infty}a(n)n^{-s}. \tag{0.15}$$

We can now state the Theorem.

THEOREM: *Let φ be an even Hecke normalized Maass cusp form in U then*

(i) *If $d < 0$*

$$\frac{1}{\langle\varphi,\varphi\rangle}\sum_{z\in\Lambda_d}{}^{*}\varphi(z) = 24\pi|d|^{3/4}\sum_{\mathrm{Shim}(F_j)=\varphi}\rho_j(d)\overline{\rho_j(1)}. \tag{0.16}$$

(ii) *If $d > 0$*

$$\frac{1}{\langle\varphi,\varphi\rangle}\sum_{C\in\Lambda_d}\int_C\varphi ds = 12\pi^{1/2}d^{3/4}\sum_{\mathrm{Shim}(F_j)=\varphi}\rho_j(d)\overline{\rho_j(1)}, \tag{0.17}$$

where ds is the hyperbolic arc length.

Some remarks concerning the Theorem are the following.

(a) The sum on the right hand side (hence-forth the spectral side) is finite. It is over an orthonormal basis of the finite dimensional subspace of V^+ consisting of F's whose T_{p^2} eigenvalues and $\Delta_{1/2}$ eigenvalue correspond via (0.15) to those Hecke eigenvalues T_p ($p \neq 2$) of φ and Laplace eigenvalue. The sum in question is clearly independent of the orthonormal basis chosen. One could check using the trace formula whether the space of such F_j's is 1-dimensional, but we have not done so.

(b) In the form we have stated (0.16), it clearly gives the promised generalization of (0.2).

(c) If $d = 1$ then Λ_1 consists of the single form xy, the corresponding cycle is the geodesic $[0, i\infty]$, and (0.17) yields

$$\frac{1}{<\varphi,\varphi>}\int_0^{\infty}\varphi(iy)\frac{dy}{y} = 12\pi^{1/2}\sum_{\mathrm{Shim}(F_j)=\varphi}|\rho_j(1)|^2. \tag{0.18}$$

Hence by Proposition 1.3 with $s = 0$

$$\frac{\pi^{-1/2}\Gamma(\frac{1/2+2ir}{2})\Gamma(\frac{1/2-2ir}{2})L(\varphi,\frac{1}{2})}{<\varphi,\varphi>} = 12\pi^{1/2}\sum_{\mathrm{Shim}(F_j)=\varphi}|\rho_j(1)|^2 \tag{0.19}$$

where $L(\varphi, s)$ is the L-function of φ defined in (0.13). In particular, we have

COROLLARY 1: *If φ is an even Hecke normalized eigenform in U then*

$$L(\varphi, \frac{1}{2}) \geq 0.$$

Note that if φ is odd then $L(\varphi, \frac{1}{2}) = 0$ simply because of the sign of the functional equation.

(d) If $L(\varphi, \frac{1}{2}) = 0$ then the sum in the right-hand side of (0.19) is vacuous, hence we have

COROLLARY 2: *If $L(\varphi, \frac{1}{2}) = 0$ then*

$$\sum_{z \in \Lambda_d} \varphi(z) = 0 \quad \textit{for all } d < 0,$$
(0.20)
$$\sum_{C \in \Lambda_d} \int_C \varphi dt = 0 \quad \textit{for all } d < 0.$$

Such cusp forms, if they exist, would lie in the space introduced by Zagier [Z, (21)].

(e) In the holomorphic setting the analogue of the Theorem has been developed in much greater generality, specifically, on the right-hand side two general coefficients $\rho_j(d)\overline{\rho_j(d')}$ are allowed as well as forms on congruence subgroups $\Gamma_0(N)$, see [Ko2], [S] and Shimura's recent paper [S2] where the holomorphic version of (0.19) is derived in great generality including Hilbert modular forms. It would be interesting to give such extensions of the above Theorem. Our interest in this formula, however, was primarily in connection with the class numbers $h(d) = |\Lambda_d|$ (and the corresponding location of these C.M. points) which is the case captured by the Theorem.

An outline of the paper is as follows. In §1 we review some results about Maass forms and Hecke operators. In §2 we introduce the Theta function of Siegel which is central in our analysis. In §3 we compute the left hand side of (0.16) and (0.17) (hence-forth known as the geometric side). In §4 the key proportionality constant on the spectral side is derived using a technique of Niwa. Finally, in §5 we deduce the identity.

1. Background: Maass cusp forms and Hecke operators

In this section we review some facts about Maass cusp forms and Hecke operators. The Hecke operators T_p: $U \to U$ are defined by the following formula for all primes p:

$$T_p\varphi(z) = \sum_{n\neq 0}\{p^{1/2}b(np) + p^{-1/2}b\Big(\frac{n}{p}\Big)\}W_{0,ir}(4\pi|n|y)e(nx). \tag{1.1}$$

(As usual, here and further we assume that $b(t) = 0$ if t is not an integer.) The following propositions follow as in the holomorphic setting, see e.g. [T], [AL].

PROPOSITION 1.1:

(i) $\{T_p\}$ *is a commuting family of linear self-adjoint operators* $T_p\colon U \to U$.

(ii) T_p *coincide with the Hecke operators introduced by Maass:*

$$T_p\varphi(z) = p^{-1/2}\Big(\sum_{j=0}^{p-1}\varphi\Big(\frac{z+j}{p}\Big) + \varphi(pz)\Big).$$

(iii) T_p *commute with the Laplacian* Δ_0.

(iv) *There exists a basis of* U, $\{\varphi_j\}$, *such that* $T_p\varphi_j = \lambda_j(p)\varphi_j$, $\Delta_0\varphi_j = \lambda_j(0)\varphi_j$.

We shall refer to the following result as the Strong Multiplicity Theorem for $SL_2(\mathbb{Z})$:

PROPOSITION 1.2: *Let* U_0 *be a subspace of* U *consisting of eigenfunctions of* Δ_0 *and* T_p *for all but finitely many* p. *Then* $\dim(U_0) = 1$. *Moreover, if* $\varphi \in U_0$, φ *is an eigenfunction for all* T_p. *If* $\varphi(z)$ *has the Fourier development*

$$\varphi(z) = \sum_{n\neq 0} b(n)W_{0,ir}(4\pi|n|y)e(nx), \tag{1.2}$$

then $b(1) = 0$ *if and only if* $\varphi = 0$.

PROPOSITION 1.3: *If* φ *is an even Maass cusp form with eigenvalue equal to* $-\frac{1}{4} - (2r)^2$, *then*

$$\Omega(s) \triangleq \int_0^\infty \varphi(iy)y^s\frac{dy}{y} = \pi^{-s-1/2}\Gamma(\frac{s+1/2+2ir}{2})\Gamma(\frac{s+1/2-2ir}{2})\cdot L(\varphi, s+\frac{1}{2}),$$

$\Omega(s)$ *is entire and* $\Omega(1-s) = \Omega(s)$.

For each prime $p \neq 2$ introduce the Hecke operator $T_{p^2}\colon V \to V$ by the formula

$$\begin{aligned} T_{p^2}F(z) = \sum_{n\neq 0}\Big\{&pb(np^2) + p^{-1/2}\Big(\frac{n}{p}\Big)b(n) \\ &+ p^{-1}b\Big(\frac{n}{p^2}\Big)\Big\}W_{\frac{1}{4}\mathrm{sgn}(n),ir}(4\pi|n|y)e(nx), \end{aligned} \tag{1.3}$$

where $\left(\frac{n}{p}\right)$ is the quadratic residue symbol. Notice that this formula is consistent with [S] and [Ko1] if we take into account the difference between our definition of the space V and a definition of a space of cusp forms of $\frac{1}{2}$-integral weight usually used in the holomorphic case [S]. The following proposition establishes some properties of the Hecke operators T_{p^2} and the operator $L = \tau_2\sigma$ introduced in the Introduction (see (0.7) and (0.8)); it follows from [S], [N1] and [Ko1].

PROPOSITION 1.4:

(i) *For $p \neq 2$, T_{p^2} is a commuting family of linear self-adjoint operators $T_{p^2}\colon V \to V$.*

(ii) *T_{p^2} commute with $\Delta_{1/2}$.*

(iii) *L is self-adjoint and satisfies the equation $(L-1)(L+\frac{1}{2}) = 0$.*

(iv) *T_{p^2} commute with L.*

2. Θ-liftings

We start with construction of Θ-functions through the Weil representation [Sh], [N]. Let R be an n-dimensional real vector space, and Q be an $n \times n$ rational symmetric matrix with $p > 0$ positive and $q = n - p$ negative eigenvalues. Take a lattice L in R and denote by L^* the lattice dual to L in R:

$$L^* = \{x \in R \mid \langle x, y\rangle = {}^t x Q y \in \mathbb{Z} \quad \text{for any } y \in L\}.$$

We assume $L^* \supset L$.

For

$$\sigma = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{SL}_2(\mathbb{R})$$

and $f(x)$ in $L^2(R)$, the Weil representation $\sigma \to r_0(\sigma)$ is defined as follows:

$$(2.1)\qquad (r_0(\sigma)f)(x) = \begin{cases} |a|^{n/2} e[\frac{ab}{2} < x, x >] f(ax) & \text{for } c = 0, \\ |c|^{-n/2}\sqrt{|\det Q|} \int_R e[\frac{a<x,x>-2<x,y>+d<y,y>}{2c}] f(y)\,dy & \text{for } c \neq 0, \end{cases}$$

where $e(x) = \exp(2\pi i x)$ as usual.

For any Schwartz function $f \in \mathcal{S}(R)$ and any $k \in L^*/L$ we define

$$\theta(f,k) = \sum_{x \in L} f(x+k).$$

For $z = u + iv \in \mathfrak{H}$, let

$$\sigma_z = \begin{bmatrix} \sqrt{v} & u/\sqrt{v} \\ 0 & 1/\sqrt{v} \end{bmatrix},$$

and

$$\theta(z, f, k) = v^{-1/4}\theta(r_0(\sigma_z)f, k). \tag{2.2}$$

PROPOSITION 2.1 ([Sh]): *Let*

$$\gamma \in \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{SL}_2(\mathbb{Z})$$

such that

$$ab\langle x, x\rangle \equiv cd\langle x, x\rangle \equiv 0 \pmod 2 \quad \textit{for all } x \in L, \tag{2.3}$$

and suppose f satisfies $\varepsilon(k(\theta))r_0(k(\theta))f = (\cos\theta - i\sin\theta)^{-\kappa/2} f$ for all θ, where

$$k(\theta) = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix},$$

κ is a positive integer, and

$$\varepsilon(k(\theta)) = \begin{cases} \sqrt{i}^{\,\mathrm{sgn}(-\sin\theta)} & \textit{if } \theta \neq 0, \\ \sqrt{i}^{\,(1-\mathrm{sgn}\cos\theta)} & \textit{if } \theta = 0. \end{cases}$$

Then if $c \neq 0$

$$\sqrt{i}^{\,(p-q)\mathrm{sgn}\,c}(cz+d)^{-\kappa/2}\theta(\gamma z, f, h) = \sum_{k \in L^*/L} c(h,k)_\gamma \theta(z, f, k)$$

where

$$c(h,k)_\gamma = \sqrt{|\det Q|}^{-1}\mathrm{vol}(L)^{-1}|c|^{-n/2} \cdot \sum_{r \in L/cL} e\left[\frac{a\langle h+r, h+r\rangle - 2\langle k, h+r\rangle + d\langle k,k\rangle}{2c}\right].$$

We apply this Proposition to obtain transformation formulae for certain θ-functions.

(1) $\theta_1(z) = \sum_{n=-\infty}^{\infty} e(n^2 z)$.

Let $n=1$, $p=1$, $q=0$; $Q=2$, $\langle x,y\rangle = 2xy$; $L=\mathbb{Z}$; $f_1(x)=\exp(-2\pi x^2)$. Then $L^*=\mathbb{Z}/2$, $\kappa=1$, and $(r_0(\sigma_z)f_1)(x) = v^{1/4}\exp(2\pi i x^2 z)$. We have

$$\theta(z,f_1,0)=\theta(r_0(\sigma_z)f_1,0)=v^{-1/4}\sum_{x\in L} r_0(\sigma_z)f_1(x+0)=\sum_{x\in\mathbb{Z}} e(x^2z)=\theta_1(z),$$

$$\begin{aligned}\theta(z,f_1,\frac{1}{2})=\theta(r_0(\sigma_z)f_1,\frac{1}{2})&=v^{-1/4}\sum_{x\in L} r_0(\sigma_z)f_1(x+\frac{1}{2})\\&=\sum_{x\in\mathbb{Z}} e((x+\frac{1}{2})^2z)=\theta_2(z).\end{aligned}$$

Let

$$\gamma_2=\begin{bmatrix}0&-1\\1&0\end{bmatrix}.$$

It satisfies (2.3), $\det Q=2$, and $\mathrm{vol}(L)=1$. Proposition 2.1 with $f=f_1$ yields

$$\sqrt{i}z^{-1/2}\theta(\gamma_2z,f_1,0)=\sum_{k\in L^*/L} c(0,k)_{\gamma_2}\theta(z,f_1,k)=\frac{1}{\sqrt{2}}(\theta(z,f_1,0)+\theta(z,f_1,\frac{1}{2})),$$

since $c(0,k)_{\gamma_2}=\frac{1}{\sqrt{2}}$, and

$$\sqrt{i}z^{-1/2}\theta(\gamma_2z,f_1,\frac{1}{2})=\sum_{k\in L^*/L} c(\frac{1}{2},k)_{\gamma_2}\theta(z,f_1,k)=\frac{1}{\sqrt{2}}(\theta(z,f_1,0)-\theta(z,f_1,\frac{1}{2})),$$

since $c(\frac{1}{2},0)_{\gamma_2}=\frac{1}{\sqrt{2}}$, and $c(\frac{1}{2},\frac{1}{2})_{\gamma_2}=-\frac{1}{\sqrt{2}}$. These relations can be rewritten in terms of θ_1 and θ_2 as follows:

$$\begin{aligned}\theta_1(\gamma_2z)&=i^{-1/2}z^{1/2}2^{-1/2}(\theta_1(z)+\theta_2(z)),\\ \theta_2(\gamma_2z)&=i^{-1/2}z^{1/2}2^{-1/2}(\theta_1(z)-\theta_2(z)).\end{aligned}$$

The transformation formulae under

$$\tau_2=\begin{bmatrix}0&-\frac{1}{2}\\2&0\end{bmatrix}$$

derived below, will be used in §4. We have

$$\tau_2=\begin{bmatrix}0&-1\\1&0\end{bmatrix}\begin{bmatrix}2&0\\0&\frac{1}{2}\end{bmatrix},$$

hence

$$\begin{aligned}\theta_1(\tau_2z)&=\theta_1(\gamma_2(4z))\\&=i^{-1/2}(4z)^{1/2}2^{-1/2}\Big(\sum_{x\in\mathbb{Z}}\exp(2\pi ix^2\cdot 4z)+\sum_{x\in\mathbb{Z}}\exp(2\pi i(x+\frac{1}{2})^2\cdot 4z)\Big)\\&=i^{-1/2}z^{1/2}2^{1/2}\Big(\sum_{x\in\mathbb{Z}}\exp(2\pi i(2x)^2z)+\sum_{x\in\mathbb{Z}}\exp(2\pi i(2x+1)^2z)\Big)\\&=i^{-1/2}z^{1/2}2^{1/2}\sum_{x\in\mathbb{Z}}\exp(2\pi ix^2z)\end{aligned}$$

and we get

$$\theta_1(\tau_2 z) = \theta_1(z) i^{-1/2} z^{1/2} 2^{1/2}.$$

For $\theta(z) = v^{1/4}\theta_1(z)$ we obtain the formula

$$\theta(\tau_2 z) = \theta(z) e^{-i\pi/4} \left(\frac{z}{|z|}\right)^{1/2},$$

which can be rewritten using the notation (0.7) as follows:

$$(\tau_2\theta)(z) = \theta(z) = v^{1/4}(\theta_1(4z) + \theta_2(4z)). \tag{2.4}$$

Similarly,

$$\begin{aligned}\theta_2(\tau_2 z) &= \theta_2(\gamma_2(4z)) \\ &= i^{-1/2}(4z)^{1/2}2^{-1/2}(\sum_{x\in\mathbb{Z}} \exp(2\pi i x^2 \cdot 4z) - \sum_{x\in\mathbb{Z}} \exp(2\pi i (x + \frac{1}{2})^2 \cdot 4z)) \\ &= i^{-1/2} z^{1/2} 2^{1/2} \Big(\sum_{x\in\mathbb{Z}} \exp(2\pi i (2x)^2 z) - \sum_{x\in\mathbb{Z}} \exp(2\pi i (2x+1)^2 z)\Big),\end{aligned}$$

and we get

$$\theta_2(\tau_2 z) = i^{-1/2} z^{1/2} 2^{1/2} (\theta_1(4z) - \theta_2(4z)).$$

For $\theta_{1/2}(z) = v^{1/4}\theta_2(z)$ we obtain

$$(\tau_2\theta_{1/2})(z) = v^{1/4}(\theta_1(4z) - \theta_2(4z)). \tag{2.5}$$

(2) Let $n = 3$, $p = 2$, $q = 1$,

$$Q = 2\begin{bmatrix} 0 & 0 & -2 \\ 0 & 1 & 0 \\ -2 & 0 & 0 \end{bmatrix},$$

$\langle x, y\rangle = {}^t x Q y$; $L = \mathbb{Z}\oplus\mathbb{Z}\oplus\mathbb{Z}$, $f_3(x_1, x_2, x_3) = \exp(-2\pi(2x_1^2 + x_2^2 + 2x_3^2))$. Then $L^* = \frac{\mathbb{Z}}{4}\oplus\frac{\mathbb{Z}}{2}\oplus\frac{\mathbb{Z}}{4}$ and $\kappa = 1$.

The group $\mathrm{SL}_2(\mathbb{R})$ acts on R by $g(x_1, x_2, x_3) = (x_1', x_2', x_3')$ where

$$\begin{bmatrix} x_1' & x_2'/2 \\ x_2'/2 & x_3' \end{bmatrix} = g \begin{bmatrix} x_1 & x_2/2 \\ x_2/2 & x_3 \end{bmatrix} {}^t g, \tag{2.6}$$

and on $\mathcal{S}(\mathbb{R}^3)$ by $(gf)(x) = f(g^{-1}(x))$.

For $z \in \mathfrak{H}$ and $g \in \mathrm{SL}_2(\mathbb{R})$ we define

$$\theta(z,g,f,k) = \theta(z,gf,k) \tag{2.7}$$

where the θ-function in the right hand side was defined in (2.2). Then

$$(r_0(\sigma_z)gf_3)(x) = v^{3/4}e\Big[u\frac{\langle x,x\rangle}{2}\Big]f_3(\sqrt{v}g^{-1}(x)),$$

and

$$\begin{aligned}\theta(z,g,f_3,0) &= \theta(z,gf_3,0) = v^{-1/4}\theta(r_0(\sigma_z)gf_3,0)\\ &=\sqrt{v}\sum_{x\in \mathbb{Z}^3} e[u(x_2^2-4x_1x_3)]f_3(\sqrt{v}g^{-1}(x)) = \theta(z,g).\end{aligned}$$

The Θ-function introduced above is in fact Siegel's Θ-function [Si]. More precisely, let

$$S = \begin{bmatrix} 0 & 0 & -2\\ 0 & 1 & 0\\ -2 & 0 & 0\end{bmatrix},$$

and $\Omega(S)$ be the set of all real 3×3 matrices C such that

$$S[C] \triangleq {}^tCSC = S.$$

Since S has two positive and one negative eigenvalue, the group $\Omega(S) \approx \mathrm{SO}(2,1)$, and it can be represented by 3×3 matrices as follows:

$$\Omega(S) = \left\{ \begin{bmatrix} a^2 & ab & b^2\\ 2ac & ad+bc & 2bd\\ c^2 & cd & d^2\end{bmatrix},\ \text{where } \begin{bmatrix} a & b\\ c & d\end{bmatrix} \in \mathrm{SL}_2(\mathbb{R})\right\}. \tag{2.8}$$

Let $\mathcal{H}$ be the representation space of $\Omega(S)$:

$$\mathcal{H} = \{{}^tH = H,\ H>0,\ HS^{-1}H = S\},$$

where $\Omega(S)$ acts by $H \to H[C]$.

Let $z = u+iv \in \mathfrak{H}$, $H \in \mathcal{H}$, and $R = uS+ivH$. Following Siegel [Si], we define

$$\theta(z,H) = \sqrt{v}\sum_{h\in\mathbb{Z}^3} e[R[h]] = \sqrt{v}\sum_{h\in\mathbb{Z}^3} e^{2\pi i\, {}^thRh}.$$

Let

$$H_0 = \begin{bmatrix} 2 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 2\end{bmatrix}.$$

It is easily seen that $H_0S^{-1}H_0 = S$, i.e. $H_0 \in \mathcal{H}$. $\Omega(S)$ acts transitively on $\mathcal{H}$, hence any $H \in \mathcal{H}$ can be written as $H = H_0[C]$ for some $C \in \Omega(S)$. Taking into account the isomorphism between SO(2, 1) and $\mathrm{PSL}_2(\mathbb{R})$ given by (2.8) we define for $z \in \mathfrak{H}$ and $g \in \mathrm{SL}_2(\mathbb{R})$ (we slightly abuse notations using the same letter g for the corresponding elements of $\mathrm{SL}_2(\mathbb{R})$ and SO(2, 1))

$$\theta(z,g) = \theta(z, H_0[g^{-1}]) = \sqrt{v} \sum_{h\in\mathbb{Z}^3} e[u(h_2^2 - 4h_1h_3)] f_3(\sqrt{v}g^{-1}(h)),$$

where $f_3(h_1, h_2, h_3) = \exp(-2\pi(2h_1^2 + h_2^2 + 2h_3^2))$.

Now let

$$\Theta(z,g) = v^{1/4}\theta(z,g) = v^{3/4} \sum_{h\in\mathbb{Z}^3} e[u(h_2^2 - 4h_1h_3)] f_3(\sqrt{v}g^{-1}(h)). \tag{2.9}$$

PROPOSITION 2.2:

(i) $\Theta(\gamma z, g) = J(\gamma, z)\Theta(z,g)$ *for any* $\gamma \in \Gamma_0(4)$.

(ii) $\Theta(z, \gamma g k) = \Theta(z,g)$ *for any* $\gamma \in \mathrm{SL}_2(\mathbb{Z})$ *and*

$$k \in K = \left\{ \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}, 0 \le \theta < 2\pi \right\}.$$

(iii) $\Theta\left(z, \begin{bmatrix} 1 & \xi \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \eta^{1/2} & 0 \\ 0 & \eta^{-1/2} \end{bmatrix}\right) = \Theta\left(z, \begin{bmatrix} 1 & -\xi \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \eta^{1/2} & 0 \\ 0 & \eta^{-1/2} \end{bmatrix}\right)$, i.e. Θ *is even in* ξ.

Proof: (i) This follows from Proposition 2.1 with $\kappa = 1$, see e.g. [N, D].

(ii) Let $\gamma \in \mathrm{SL}_2(\mathbb{Z})$. Then

$$\Theta(z, \gamma g) = v^{3/4} \sum_{h\in\mathbb{Z}^3} e[u(h_2^2 - 4h_1h_3)] f_3(\sqrt{v}g^{-1}(\gamma^{-1}h)).$$

Let $x = \gamma^{-1}h$. Then $h = \gamma x$ and $h_2^2 - 4h_1h_3 = {}^thSh = {}^tx\,{}^t\gamma S\gamma x = x\,{}^tSx = x_2^2 - 4x_1x_3$ since $\gamma \in \Omega(S)$. Since $x \in \mathbb{Z}^3$ we can rewrite

$$\Theta(z, \gamma g) = v^{3/4} \sum_{x\in\mathbb{Z}^3} e[u(x_2^2 - 4x_1x_3)] f_3(\sqrt{v}g^{-1}(x)) = \Theta(z,g). \qquad \blacksquare$$

While $\Theta(z,g)$ is not an eigenform of the Laplacian in either z or g, it does satisfy the fundamental relation (see [Sh], [N])

$$D_g\Theta(z,g) = 4\Delta_{1/2}\Theta(z,g) + \frac{3}{4}\Theta(z,g) \tag{2.10}$$

where D_g is the Casimir operator on $\mathrm{SL}_2(\mathbb{R})$, appropriately normalized (see [GGP], §1). This relation is responsible for Proposition 2.3 below.

A Maass–Hecke form φ can be viewed as being defined on $G = \mathrm{SL}_2(\mathbb{R})$ and K-invariant on the right where

$$K = \left\{ \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}, \quad 0 \le \theta < 2\pi \right\}.$$

φ in this setting is an eigenfunction of the Casimir operator D_g:

$$D_g\varphi(g) = (-\frac{1}{4} - (2r)^2)\varphi(g).$$

PROPOSITION 2.3: *Let $\varphi \in U$, φ an even Maass–Hecke normalized cusp form, and*

$$F(z) = \int_{\Gamma\backslash G} \varphi(g)\Theta(z,g)dg \tag{2.11}$$

(notice that if φ is odd then $F \equiv 0$ since Θ is even by Proposition 2.2 (iii)). Then

(i) *F is an eigenfunction of $\Delta_{1/2}$.*

(ii) *$F \in V^+$.*

Proof: (i) By integration by parts we have

$$\begin{aligned}(-\frac{1}{4} - (2r)^2)F(z) &= \int_{\Gamma\backslash G} D_g\varphi(g)\Theta(z,g)dg = \int_{\Gamma\backslash G} \varphi(g)D_g\Theta(z,g)dg \\ &= \int_{\Gamma\backslash G} 4\varphi(g)\Delta_{1/2}\Theta(z,g)dg + \int_{\Gamma\backslash G} \varphi(g)\frac{3}{4}\Theta(z,g)dg \\ &= 4\Delta_{1/2}F(z) + \frac{3}{4}F(z),\end{aligned}$$

and hence

$$\Delta_{1/2}F(z) = (-\frac{1}{4} - r^2)F(z). \tag{2.12}$$

Notice that the eigenvalues of φ and F are not equal but related.

(ii) First we show that $F \in V$. The transformation formula follows from the transformation formula for Θ (see Proposition 2.2 (i)). We check that $F(z)$ is cuspidal at the cusp at ∞ (the other cusps are dealt with by similar argument). Since F is an eigenfunction of $\Delta_{1/2}$, it suffices to show that $F(u + iv)$ is rapidly decreasing as $v \to \infty$. Now examination of the series (2.9) defining $\Theta(z,g)$ shows that

$$\Theta(z,g) - v^{3/4} f_3(0)$$

is rapidly decreasing as $v \to \infty$ (all terms in the series with $h \neq 0$ are rapidly decreasing). Since φ is a cusp form we can infer the same after integration against $\varphi(g)$. Also

$$\int_{\Gamma\backslash G} \varphi(g) v^{3/4} f_3(0) dg = 0$$

since φ is a Maass cusp form. Thus $F(z)$ is rapidly decreasing at the cusp at ∞.

Let

$$M_n(v) = \int_0^1 \Big(\int_{\Gamma\backslash G} \varphi(g)\Theta(u+iv,g)dg \Big) e(-nu)du. \tag{2.13}$$

Clearly

$$M_n(v) = \int_{\Gamma\backslash G} \sum_{x_2^2 - 4x_1x_3 = n} v^{3/4} f_3(\sqrt{v} g^{-1} x)\varphi(g)dg. \tag{2.14}$$

Thus

$$M_n(v) = 0 \quad \text{if } n \equiv 2,3 \pmod 4. \tag{2.15}$$

It follows as in the holomorphic case [Ko1] that any Maass form from V whose Fourier coefficients satisfy (2.15) is an eigenfunction of L with eigenvalue 1, thus $F \in V^+$. ∎

3. Geometric calculation

We shall examine the behavior of

$$M_n(v) = \int_0^1 \int_{\Gamma\backslash G} \varphi(g)\Theta(u+iv,g)dg \; e(-nu)du. \tag{3.1}$$

We know from general facts that $M_n(v)$ is a Whittaker function in v—we need it more precisely. We have seen in (2.15) that $M_n(v) = 0$ if $n \equiv 2,3 \pmod 4$. Otherwise the solutions to $x_2^2 - 4x_1x_3 = n$ break up into $h(n)$ orbits under the action of Γ. Let $x_{(1)}, \ldots, x_{(h(n))}$ denote representatives for these orbits and let $\Gamma_{x_{(j)}} = \mathrm{stab}\, x_{(j)}$ in Γ. We have

$$\begin{aligned} M_n(v) &= \int_{\Gamma\backslash G} \sum_{x_2^2-4x_1x_3=n} v^{3/4} f_3(\sqrt{v} g^{-1}x)\varphi(g)dg \\ &= v^{3/4} \sum_{j=1}^{h(n)} \int_{\Gamma\backslash G} \sum_{\gamma\in\Gamma_{x_{(j)}}\backslash\Gamma} f_3(\sqrt{v} g^{-1}\gamma^{-1}x_{(j)})\varphi(g)dg \qquad (3.2) \\ &= v^{3/4} \sum_{j=1}^{h(n)} \int_{\Gamma_{x_{(j)}}\backslash G} f_3(\sqrt{v} g^{-1}x_{(j)})\varphi(g)dg. \end{aligned}$$

We consider now two basically different cases: in the first $n < 0$ and in the second $n > 0$.

CASE (i): $n < 0$. Let $x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ be such that $x_2^2 - 4x_1x_3 = n$ and

$$I = I(x) = \int_G f_3(\sqrt{v} g^{-1}x)\varphi(g)dg.$$

Then (3.2) becomes

$$M_n(v) = v^{3/4} \sum_{j=1}^{h(n)} \frac{1}{|\Gamma_{x_{(j)}}|} I(x_{(j)}) = v^{3/4} \sum_{x\in\Lambda_n}{}^{*} I(x). \qquad (3.3)$$

$G \approx \mathrm{SL}_2(\mathbb{R})$ acts transitively on the hyperboloid ${}^t xSx = n < 0$, so we can find a $g_1 \in G$ such that

$$g_1^{-1}x = \frac{\sqrt{|n|}}{2} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}. \qquad (3.4)$$

Thus

$$\begin{aligned} I &= \int_G f_3(\sqrt{v}(g_1g)^{-1}x)\varphi(g_1g)dg \\ &= \int_G f_3\left(\sqrt{\frac{v|n|}{4}} g^{-1} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}\right)\psi(g)dg \end{aligned}$$

where

$$\psi(g) = \varphi(g_1g). \qquad (3.5)$$

Now

$$f_3\left(\sqrt{\frac{v|n|}{4}}g^{-1}\begin{bmatrix}1\\0\\1\end{bmatrix}\right)=f_3\left(\sqrt{\frac{v|n|}{4}}(k_2gk_1)^{-1}\begin{bmatrix}1\\0\\1\end{bmatrix}\right). \tag{3.6}$$

Hence using the Cartan decomposition $G = KA^+K$ we have

$$I=\int_1^\infty f_3\left(\sqrt{\frac{v|n|}{4}}\begin{bmatrix}a^{-2}&0&0\\0&1&0\\0&0&a^2\end{bmatrix}\begin{bmatrix}1\\0\\1\end{bmatrix}\right)\left(\int_K\int_K\psi(k_1gk_2)dk_1dk_2\right)\cdot\delta(a)\frac{da}{a}$$

where $\delta(a)=\frac{a^2-a^{-2}}{2}$ (see [L]).

Now from (3.5), $\psi(g)$ is an eigenfunction of the Casimir operator with the same eigenvalue λ as φ, hence

$$\tilde{\psi}(g)=\int_K\int_K\psi(k_1gk_2)dk_1dk_2$$

is a spherical function (see [L]). So by a standard Uniqueness Theorem

$$\tilde{\psi}(g)=\tilde{\psi}(e)\omega_\lambda(g) \tag{3.7}$$

where $\omega_\lambda(e)=1$ and $\omega_\lambda(g)$ is the standard spherical function with eigenvalue λ. Clearly from (3.5)

$$\tilde{\psi}(e)=\psi(e)=\varphi(g_1). \tag{3.8}$$

Thus

$$I=\varphi(g_1)Y_\lambda\left(\sqrt{\frac{v|n|}{4}}\right) \tag{3.9}$$

where

$$Y_\lambda(t)=\int_1^\infty f_3\left(t\begin{bmatrix}a^{-2}\\0\\a^2\end{bmatrix}\right)\omega_\lambda\left(\begin{bmatrix}a&0\\0&a^{-1}\end{bmatrix}\right)\delta(a)\frac{da}{a}. \tag{3.10}$$

If $cx^2+bxy+ay^2$ is an integral quadratic form of discriminant n corresponding to the vector

$$x=\begin{bmatrix}c\\b\\a\end{bmatrix}$$

then it is easily verified that

$$g_1 = \begin{bmatrix} 1 & b/2a \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} (\frac{\sqrt{|n|}}{2a})^{1/2} & 0 \\ 0 & (\frac{\sqrt{|n|}}{2a})^{-1/2} \end{bmatrix}$$

satisfies (3.4). Hence

$$\varphi(g_1) = \varphi(w) \tag{3.11}$$

where $w = \zeta + i\eta \in \mathfrak{H}$ is the Heegner point corresponding to the form $ax^2 - bxy + cy^2$ which is Γ-equivalent to $cx^2 + bxy + ay^2$. Thus we have

$$M_n(v) = v^{3/4} Y_\lambda\Big(\sqrt{\frac{v|n|}{4}}\Big) \sum_{z \in \Lambda_n}{}^{*} \varphi(z). \tag{3.12}$$

For later comparison we need only determine the asymptotics as $v \to \infty$ of $Y_\lambda\Big(\sqrt{\frac{v|n|}{4}}\Big)$. In this way we can avoid evaluating the integral in (3.10). We have

$$\begin{aligned} Y_\lambda(t) &= \int_1^\infty f_3\left(t \begin{bmatrix} a^{-2} \\ 0 \\ a^2 \end{bmatrix}\right) \omega_\lambda\Big(\begin{bmatrix} a & 0 \\ 0 & a^{-1} \end{bmatrix}\Big)\delta(a)\frac{da}{a} \\ &= \int_0^\infty e^{-2\pi(2t^2a^{-4}+2t^2a^4)} \omega_\lambda\Big(\begin{bmatrix} a & 0 \\ 0 & a^{-1} \end{bmatrix}\Big)\delta(a)\frac{da}{a} \\ &= \int_0^\infty e^{-8\pi t^2 \cosh(2u)} \omega_\lambda(u) \sinh(u) du \sim \int_0^\infty e^{-8\pi t^2(1+2u^2)} u du \quad \text{as } t \to \infty \\ &= e^{-8\pi t^2} \int_0^\infty e^{-16\pi t^2 u^2} u du = \frac{e^{-8\pi t^2}}{32\pi t^2}. \end{aligned}$$

Thus

$$M_n(v) \sim \frac{v^{-1/4}}{8\pi|n|} e^{-2\pi v|n|} \Big(\sum_{z \in \Lambda_n}{}^{*} \varphi(z)\Big) \quad \text{as } v \to \infty. \tag{3.13}$$

Since we know $M_n(v) = \rho(n) W_{-1/4, ir}(4\pi|n|v)$, we have

$$M_n(v) \sim \rho(n) e^{-2\pi|n|v} v^{-1/4} \tag{3.14}$$

which is consistent with (3.13).

CASE (ii): $n > 0$. Consider

$$I = \int_{\Gamma_{x_{(j)}}\backslash G} f_3(\sqrt{v}g^{-1}x_{(j)})\varphi(g)dg.$$

This time we can find $h \in G$ such that

$$(3.15)\qquad \begin{aligned} h^{-1}x_{(j)} &= \begin{bmatrix} 0 \\ \sqrt{n} \\ 0 \end{bmatrix}, \\ I &= \int_{\Gamma_{x'_{(j)}}\backslash G} f_3\left(\sqrt{v}g^{-1}\begin{bmatrix} 0 \\ \sqrt{n} \\ 0 \end{bmatrix}\right)\varphi(hg)dg, \end{aligned}$$

where

$$(3.16)\qquad \Gamma_{x'_{(j)}} = h^{-1}\Gamma_{x_{(j)}}h.$$

$\Gamma_{x'_{(j)}}$ is a discrete subgroup of the stabilizer of $\begin{bmatrix} 0 \\ \sqrt{n} \\ 0 \end{bmatrix}$, i.e. of $\left\{\pm\left[\begin{smallmatrix} a & 0 \\ 0 & a^{-1}\end{smallmatrix}\right], a > 0\right\}$. There are two cases depending on whether $\Gamma_{x_{(j)}}$ is infinite cyclic or essentially trivial: $\Gamma_{x_{(j)}} = \{\pm 1\}$ (the first happens if n is not a square, the second if n is a square). In the first case we have

$$(3.17)\qquad \Gamma_{x'_{(j)}} = \left\{ \pm\begin{bmatrix} \varepsilon & 0 \\ 0 & \varepsilon^{-1}\end{bmatrix}^m \,\middle|\, m \in \mathbb{Z}\right\}.$$

If we write

$$(3.18)\qquad g = \begin{bmatrix} 1 & \xi \\ 0 & 1 \end{bmatrix}\begin{bmatrix} \eta^{1/2} & 0 \\ 0 & \eta^{-1/2}\end{bmatrix}k$$

then a fundamental domain for $\Gamma_{x'_{(j)}}\backslash G$ is $\{-\infty < \xi < \infty,\ 1 \le \eta \le \varepsilon^2\}$. Then I becomes

$$(3.19)\qquad \begin{aligned} I &= \int_1^{\varepsilon^2}\int_{-\infty}^{\infty} f_3\left(\sqrt{vn}\begin{bmatrix} -\xi/\eta \\ 1 \\ 0 \end{bmatrix}\right)\varphi\left(h\begin{bmatrix} \eta^{1/2}\xi & \eta^{-1/2} \\ 0 & \eta^{-1/2}\end{bmatrix}\right)\frac{d\xi d\eta}{\eta^2} \\ &= \int_1^{\varepsilon^2}\int_{-\infty}^{\infty} e^{-2\pi vn(2t^2+1)}\varphi\left(h\begin{bmatrix} \eta^{1/2} & 0 \\ 0 & \eta^{-1/2}\end{bmatrix}\begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}\right)\frac{dtd\eta}{\eta}. \end{aligned}$$

Let

$$\psi(g) = \varphi(hg). \tag{3.20}$$

ψ is an eigenfunction of the Casimir operator and is right K-invariant. Also

$$\psi\left(\begin{bmatrix} \varepsilon & 0 \\ 0 & \varepsilon^{-1} \end{bmatrix} g\right) = \psi(g). \tag{3.21}$$

Let

$$H(g) = \int_1^{\varepsilon^2} \int_K \psi\left(\begin{bmatrix} \eta^{1/2} & 0 \\ 0 & \eta^{-1/2} \end{bmatrix} gk\right) \frac{d\eta}{\eta} dk. \tag{3.22}$$

Then

$$H\left(\begin{bmatrix} a & 0 \\ 0 & a^{-1} \end{bmatrix} gk\right) = H(g), \quad a > 0, \quad k \in K.$$

Thus H is uniquely determined by its values on $\left[\begin{smallmatrix} 1 & t \\ 0 & 1 \end{smallmatrix}\right]$ in the decomposition $\left[\begin{smallmatrix} a & 0 \\ 0 & a^{-1} \end{smallmatrix}\right] \left[\begin{smallmatrix} 1 & t \\ 0 & 1 \end{smallmatrix}\right] k$. It also satisfies a second order differential equation in t. So in fact

$$H(g) = H(e)V_\lambda(g) + U_\lambda(g) \tag{3.23}$$

where $V_\lambda(g)$ is even in t and U_λ is odd, and V_λ is normalized by $V_\lambda(e) = 1$ ($U_\lambda(e) = 0$). In fact being even in t, $V_\lambda(e) = 1$ uniquely determines $V_\lambda(g)$. We note that $e^{-2\pi vn(2t^2+1)}$ is even in t and hence

$$I = H(e) \int_{-\infty}^{\infty} e^{-2\pi vn(2t^2+1)} V_\lambda\left(\begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}\right) dt \tag{3.24}$$

and

$$H(e) = \int_1^{\varepsilon^2} \varphi\left(h \begin{bmatrix} \eta^{1/2} & 0 \\ 0 & \eta^{-1/2} \end{bmatrix}\right) \frac{d\eta}{\eta}. \tag{3.25}$$

From (3.15) it is easy to see that the last is

$$\int_{C_j} \varphi ds \tag{3.26}$$

where C_j is the geodesic cycle in $\Gamma\backslash\mathfrak{H}$ corresponding to the form $x_1x^2+x_2xy+x_3y^2$ and ds is the hyperbolic arc lengh. Thus in this case we have from (3.2) and (3.24)

$$M_n(v) = v^{3/4}\left(\sum_{j=1}^{h(n)} \int_{C_j} \varphi ds\right) \cdot \int_{-\infty}^{\infty} e^{-2\pi vn(2t^2+1)} V_\lambda\left(\begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}\right) dt. \tag{3.27}$$

Again we investigate the asymptotics as $v \to \infty$.

$$\begin{aligned}\int_{-\infty}^{\infty} e^{-2\pi vn(2t^2+1)} V_\lambda(t)dt &= e^{-2\pi vn} \int_{-\infty}^{\infty} e^{-4\pi vnt^2} V_\lambda(t)dt \\ &\sim e^{-2\pi vn} \int_{-\infty}^{\infty} e^{-4\pi vnt^2} dt \quad \text{as } v \to \infty \quad (\text{since } V_\lambda(0) = 1) \\ &= \frac{e^{-2\pi vn}}{2\sqrt{\pi vn}} \sqrt{\pi}.\end{aligned}$$

Hence

$$\begin{aligned}M_n(v) &\sim \frac{v^{1/4} e^{-2\pi vn}}{2\sqrt{n}} \Big(\sum_{j=1}^{h(n)} \int_{C_j} \varphi ds \Big) \\ &= \frac{v^{1/4} e^{-2\pi vn}}{2\sqrt{n}} \Big(\sum_{C \in \Lambda_n} \int_C \varphi ds \Big).\end{aligned} \tag{3.28}$$

Again this is consistent with the asymptotics of

$$W_{1/4\mathrm{sgn}(n), ir}(4\pi|n|v) \sim e^{-2\pi nv}(4\pi nv)^{1/4}.$$

In the case that $\Gamma_{z_{(j)}} = \{\pm 1\}$ (i.e. n is a square) (3.28) still holds and is derived in the same way except that C_j is now a complete geodesic in $\mathfrak{H}$.

4. Niwa's Lemma in the non-holomorphic case

Proposition 4.1: *Let $z = u + iv$, $w = \xi + i\eta \in \mathfrak{H}$; $F(z) \in V$ be invariant under L and be a common eigenfunction of T_{p^2}, $p \neq 2$, and $\Delta_{1/2}$ with Fourier coefficients $\rho(n)$, and $\psi(w) = \int_{\Gamma_0(4)\backslash\mathfrak{H}} F(z)\overline{\Theta(z,w)}\frac{du dv}{v^2}$, where $\Theta(z,w)$ is the Siegel's Θ-function introduced in (2.9). Then*

(i) $\psi(w) \in U$.

(ii) *It is a common eigenfunction of T_p, and Δ_0.*

(iii) *If $\rho(1) = 0$, then $\psi \equiv 0$. If $\rho(1) \neq 0$, then*

$$\psi = 3\sqrt{2}\pi^{\frac{1}{4}}\rho(1)\varphi$$

where $\varphi \in U$ is the unique normalized Maass form with Fourier expansion $2\sum_{n=1}^{\infty} a(n) W_{0,2ir}(4\pi ny)\cos(2\pi nx)$ having the same eigenvalues as ψ whose Fourier coefficients $a(n)$ are defined from the following equation:

$$\zeta(s+1)\sum_{n=1}^{\infty} \frac{\rho(n^2)}{n^{s-1/2}} = \rho(1)\sum_{n=1}^{\infty} a(n)n^{-s}.$$

Proof: By Proposition 2.2(ii), $\psi(w) = \psi(-\frac{1}{w})$. We calculate the Mellin transform of ψ:

$$\Omega(s) = \int_0^\infty \psi(i\eta)\eta^s \frac{d\eta}{\eta} = \int_0^\infty \psi(\frac{i}{\eta})\eta^s \frac{d\eta}{\eta}$$
$$= \int_0^\infty \Big(\int_{\Gamma_0(4)\backslash\mathfrak{H}} v^{1/4} F(z)\overline{\theta(z, i\eta^{-1})} \frac{dudv}{v^2} \Big) \eta^s \frac{d\eta}{\eta}.$$

Notice that $\theta(z, i\eta)$ can be represented as a product of two θ-functions:

$$\theta(z,i\eta) = \sum_{x\in\mathbf{Z}^3} v^{1/2} f_3\Big(\sqrt{v}\Big(\frac{x_1}{\eta}, x_2, \eta x_3\Big)\Big)\exp(2\pi i u(x_2^2 - 4x_1x_3))$$
$$= \Big(\sum_{x_2\in\mathbf{Z}} \exp(2\pi i(u+iv)x_2^2)\Big)$$
$$\cdot\Big(\sum_{x_1,x_3\in\mathbf{Z}} v^{1/2}\exp\big(-8\pi i u x_1 x_3 - 4\pi v\eta^2 x_3^2 - \frac{4\pi v x_1^2}{\eta^2}\big)\Big)$$
$$= \theta_1(z)\theta_2(z,\eta),$$

where $\theta_1(z) = \sum_{x_2\in\mathbf{Z}} \exp(2\pi i(u+iv)x_2^2) = \sum_{x_2\in\mathbf{Z}}\exp(2\pi i z x_2^2)$ is the usual θ-function and $\theta_2(z,\eta) = \sum_{x_1,x_3\in\mathbf{Z}} v^{1/2}\exp(-8\pi i u x_1x_3 - 4\pi v\eta^2 x_3^2 - \frac{4\pi v x_1^2}{\eta^2}))$.

We use partial Poisson summation to obtain the following equality:

$$\theta_2(z,\eta) = \frac{1}{2\eta}\sum_{x_1,x_3\in\mathbf{Z}} \exp\big(-\frac{4\pi}{v\eta^2}|x_1 z + \frac{x_3}{4}|^2\big).$$

For, let $Q(x_1,x_3) = 8iux_1x_3 + 4v\eta^2x_3^2 + \frac{4vx_1^2}{\eta^2}$. Then

$$\theta_2(z,\eta) = v^{1/2}\sum_{x_1,x_3\in\mathbf{Z}} \exp(-\pi Q(x_1,x_3)) = v^{1/2}\sum_{x_1,x_3\in\mathbf{Z}} \hat{f}(x_1,x_3),$$

where

$$\hat{f}(x_1,x_3) = \int_{-\infty}^{\infty} \exp(-\pi Q(x_1,t) - 2\pi i t x_3)dt.$$

We have $(2\sqrt{v}\eta t + \frac{2iux_1}{\sqrt{v}\eta})^2 = 4v\eta^2t^2 + 8iux_1t - \frac{4u^2x_1^2}{v\eta^2}$, hence

$$\hat{f}(x_1,x_3) = \int_{-\infty}^{\infty} \exp\big(-\pi((2\sqrt{v}\eta t + \frac{2iux_1}{\sqrt{v}\eta})^2 + \frac{4|z|^2x_1^2}{v\eta^2}) - 2\pi i t x_3\big)dt$$
$$= \exp\big(-\frac{4\pi|z|^2x_1^2}{v\eta^2}\big)\int_{-\infty}^{\infty}\exp\big(-\pi\big(t_1^2 + \frac{it_1x_3}{\sqrt{v}\eta} + \frac{2ux_1x_3}{v\eta^2}\big)\big)\frac{dt_1}{2\sqrt{v}\eta}$$

where $t_1 = 2\sqrt{v}\eta t + \frac{2iux_1}{\sqrt{v}\eta}$. We obtain then

$$\hat{f}(x_1, x_3) = \exp\Big(- \frac{4\pi(|z|^2 x_1^2 + (u/2)x_1x_3)}{v\eta^2}\Big)\frac{1}{2\sqrt{v}\eta}\int_{-\infty}^{\infty}\exp(-\pi(t^2 + \frac{itx_3}{\sqrt{v}\eta}))dt.$$

We have $(t + \frac{ix_3}{2\sqrt{v}\eta})^2 = t^2 + \frac{itx_3}{\sqrt{v}\eta} - \frac{x_3^2}{4v\eta^2}$, hence

$$\begin{aligned}\hat{f}(x_1, x_3) =&\exp\Big(- \frac{4\pi(|z|^2 x_1^2 + (u/2)x_1x_3)}{v\eta^2}\Big)\\ &\cdot\frac{1}{2\sqrt{v}\eta}\int_{-\infty}^{\infty}\exp\Big(- \pi\big(t + \frac{ix_3}{2\sqrt{v}\eta}\big)^2 + \frac{x_3^2}{4v\eta^2}\Big)dt\\ =&\frac{1}{2\sqrt{v}\eta}\exp\Big(- \frac{4\pi(|z|^2 x_1^2 + (u/2)x_1x_3 + (x_3/16))}{v\eta^2}\Big)\\ =&\frac{1}{2\sqrt{v}\eta}\exp\Big(- \frac{4\pi}{v\eta^2}\big|x_1 z + \frac{x_3}{4}\big|^2\Big).\end{aligned}$$

Thus

$$\theta_2(z,\eta) =\sqrt{v}\sum_{x_1,x_3\in\mathbf{Z}}\hat{f}(x_1,x_3) = \frac{1}{2\eta}\sum_{x_1,x_3\in\mathbf{Z}}\exp(-\frac{4\pi}{v\eta^2}|x_1 z + \frac{x_3}{4}|^2).$$

Now we can rewrite

$$\begin{aligned}\Omega(s) =& \int_0^\infty \varphi(i\eta)\eta^s\frac{d\eta}{\eta} = \int_0^\infty\Big(\int_{\Gamma_0(4)\backslash\mathfrak{H}} v^{1/4}F(z)\overline{\theta(z,i\eta^{-1})}\frac{du dv}{v^2}\Big)\eta^s\frac{d\eta}{\eta}\\ =&\frac{1}{2}\int_0^\infty\int_{\Gamma_0(4)\backslash\mathfrak{H}} v^{1/4}F(z)\overline{\theta_1(z)}\\ &\cdot\Big({\sum_{x_1,x_3\in\mathbf{Z}}}'\exp\Big(- \frac{4\pi\eta^2}{v}\big|x_1 z + \frac{x_3}{4}\big|^2\Big)\frac{du dv}{v^2}\Big)\eta^{s+1}\frac{d\eta}{\eta},\end{aligned}$$

where we used that $\langle F,\theta\rangle = 0$. (In fact, θ is unique non-cuspidal eigenform of $\Delta_{1/2}$.) Let

$$\tau = \frac{2\sqrt{\pi}\eta}{\sqrt{v}}|x_1 z + \frac{x_3}{4}|.$$

Then

$$\eta = \frac{\tau\sqrt{v}}{|x_1 z + \frac{x_3}{4}|2\sqrt{\pi}}$$

and we have

$$\Omega(s) = \frac{1}{2}\Big(\int_0^\infty e^{-\tau^2}\tau^{s+1}\frac{d\tau}{\tau}\Big)(4\pi)^{-\frac{s+1}{2}}\Big(\int_{\Gamma_0(4)\backslash\mathfrak{H}} v^{1/4}F(z)\overline{\theta_1(z)}\Big)$$
$$\cdot{\sum}'_{x_1,x_3\in\mathbf{Z}}\Big(\frac{v}{|x_1z+\frac{x_3}{4}|^2}\Big)^{\frac{s+1}{2}}\frac{dudv}{v^2}$$
$$= \frac{1}{4}\Gamma\Big(\frac{s+1}{2}\Big)(4\pi)^{-\frac{s+1}{2}}\Big(\int_{\Gamma_0(4)\backslash\mathfrak{H}} v^{1/4}F(z)\overline{\theta_1(z)}\Big)$$
$$\cdot{\sum}'_{x_1,x_3\in\mathbf{Z}}\Big(\frac{v}{|x_1z+\frac{x_3}{4}|^2}\Big)^{\frac{s+1}{2}}\frac{dudv}{v^2}$$
$$= \frac{1}{4}\Gamma\Big(\frac{s+1}{2}\Big)\cdot 4^{\frac{s+1}{2}}\pi^{-\frac{s+1}{2}}\Big(\int_{\Gamma_0(4)\backslash\mathfrak{H}} v^{1/4}F(z)\overline{\theta_1(z)}\Big)$$
$$\cdot{\sum}'_{x_1,x_3\in\mathbf{Z}}\Big(\frac{v}{|4x_1z+x_3|^2}\Big)^{\frac{s+1}{2}}\frac{dudv}{v^2}$$
$$= 2^{s-1}\Gamma\Big(\frac{s+1}{2}\Big)\pi^{-\frac{s+1}{2}}\Big(\int_{\Gamma_0(4)\backslash\mathfrak{H}} v^{1/4}F(z)\overline{\theta_1(z)}\Big)$$
$$\cdot{\sum}'_{x_1,x_3\in\mathbf{Z}}\Big(\frac{v}{|4x_1z+x_3|^2}\Big)^{\frac{s+1}{2}}\frac{dudv}{v^2}.$$

There are 3 non-holomorphic Eisenstein series on $\Gamma_0(4)$ associated to each of 3 cusps of $\Gamma_0(4)\backslash\mathfrak{H}$: $\{\infty\}$, $\{0\}$ and $\{\frac{1}{2}\}$; we denote them $E_\infty(z,s)$, $E_0(z,s)$ and $E_{\frac{1}{2}}(z,s)$, respectively. The fractional linear transformations

$$\tau_2 = \begin{bmatrix} 0 & -\frac{1}{2} \\ 2 & 0 \end{bmatrix}, \quad \tau_3 = \begin{bmatrix} 1 & -\frac{1}{2} \\ 2 & 0 \end{bmatrix}$$

act on the set of cusps as follows: $\tau_2\{0\} = \{\infty\}$, $\tau_2\{\infty\} = \{0\}$, $\tau_2\{\frac{1}{2}\} = \{\frac{1}{2}\}$, $\tau_3\{0\} = \{\infty\}$, $\tau_3\{\infty\} = \{\frac{1}{2}\}$, $\tau_3\{\frac{1}{2}\} = \{0\}$, and we have $E_0(z,s) = E_\infty(\tau_2^{-1}z,s)$, and $E_{\frac{1}{2}}(z,s) = E_\infty(\tau_3^{-1}z,s)$, where

$$E_\infty(z,s) = \sum_{\sigma\in\Gamma_\infty\backslash\Gamma_0(4)} \operatorname{Im}(\sigma z)^s.$$

Let

$$B(z,s) = {\sum}'_{x_1,x_3\in\mathbf{Z}}\Big(\frac{v}{|4x_1z+x_3|^2}\Big)^s.$$

Since $B(z,s)$ is invariant under $\Gamma_0(4)$ we have

$$B(z,s) = \lambda_\infty E_\infty(z,s) + \lambda_0 E_0(z,s) + \lambda_{\frac{1}{2}} E_{\frac{1}{2}}(z,s).$$

Let $z \to \{\infty\}$, i.e. $z = iv$, and $v \to \infty$. Then

$$B(z,s) = \sum\nolimits'_{x_1,x_3\in\mathbf{Z}} \frac{v^s}{((4x_1v)^2 + x_3^2)^s} \sim 2v^s\zeta(2s).$$

Since $E_\infty(z,s) \sim v^s$ as $z \to \{\infty\}$ and $E_\infty(z,s) \to 0$ as z approaches the cusps $\{0\}$ and $\{\frac{1}{2}\}$, we have $\lambda_\infty = 2\zeta(2s)$. Now we make change of variables $z = \tau_2 w$, and let $w = i\eta$, $\eta \to \infty$. Then asymptotically we have

$$\begin{aligned} B(\tau_2 w, s) &= \sum\nolimits'_{x_1,x_3\in\mathbf{Z}} \frac{\operatorname{Im}(\tau_2 w)^s}{|4x_1(-\frac{1}{2}/2w + x_3|^{2s}} \\ &= \sum\nolimits'_{x_1,x_3\in\mathbf{Z}} \frac{\eta^s}{|2x_1 - 2x_3 w|^{2s}} \sim 2\eta^s 2^{-2s}\zeta(2s). \end{aligned}$$

Since $E_0(z,s) = E_\infty(\tau_2^{-1}z,s) \sim (\operatorname{Im} w)^s = \eta^s$ as $\eta \to \infty$, we obtain $\lambda_0 = 2\zeta(2s)2^{-2s}$. Similarly, we obtain asymptotically that $B(\tau_3^{-1}w,s) \sim 2\eta^s 2^{-2s}\zeta(2s)$, and hence $\lambda_{\frac{1}{2}} = 2\zeta(2s)2^{-2s}$.

We can rewrite $\Omega(s)$ then as a sum of 3 terms:

$$\Omega(s) = 2^s\pi^{-\frac{s+1}{2}}\Gamma(\frac{s+1}{2})\zeta(s+1)(I_1 + I_2 + I_3)$$

where

$$\begin{aligned} I_1 &= \int_{\Gamma_0(4)\backslash\mathfrak{H}} v^{1/4}F(z)\overline{\theta_1(z)}E_\infty\big(z, \frac{s+1}{2}\big)\frac{dudv}{v^2}, \\ I_2 &= 2^{-s-1}\int_{\Gamma_0(4)\backslash\mathfrak{H}} v^{1/4}F(z)\overline{\theta_1(z)}E_0\big(z, \frac{s+1}{2}\big)\frac{dudv}{v^2}, \qquad \text{and} \\ I_3 &= 2^{-s-1}\int_{\Gamma_0(4)\backslash\mathfrak{H}} v^{1/4}F(z)\overline{\theta_1(z)}E_{\frac{1}{2}}\big(z, \frac{s+1}{2}\big)\frac{dudv}{v^2}. \end{aligned}$$

First we prove that $I_2 + I_3 = \frac{1}{2}I_2$. We have

$$I_1 = \int_{\Gamma_0(4)\backslash\mathfrak{H}} v^{1/4}F(z)\overline{\theta_1(z)}E_\infty\big(z, \frac{s+1}{2}\big)\frac{dudv}{v^2} = \langle F, \quad \theta\cdot E_\infty\rangle,$$

$$\begin{aligned} 2^{s+1}I_2 &= \int_{\Gamma_0(4)\backslash\mathfrak{H}} v^{1/4}F(z)\overline{\theta_1(z)}E_0(z, \frac{s+1}{2})\frac{dudv}{v^2} \\ &= \langle F, \theta\cdot E_0\rangle = \langle \tau_2\sigma F,\ \theta\cdot E_0\rangle = \langle F,\ \tau_2\sigma(\theta\cdot E_0)\rangle, \end{aligned}$$

and

$$\begin{aligned} 2^{s+1}I_3 &= \int_{\Gamma_0(4)\backslash\mathfrak{H}} v^{1/4}F(z)\overline{\theta_1(z)}E_{\frac{1}{2}}\big(z, \frac{s+1}{2}\big)\frac{dudv}{v^2} \\ &= \langle F, \theta\cdot E_{\frac{1}{2}}\rangle = \langle \tau_2\sigma F, \theta\cdot E_{\frac{1}{2}}\rangle = \langle F, \tau_2\sigma(\theta\cdot E_{\frac{1}{2}})\rangle. \end{aligned}$$

Hence $I_2 + I_3 = \langle F, 2^{-s-1}(\tau_2\sigma(\theta \cdot E_0) + \tau_2\sigma(\theta \cdot E_{\frac{1}{2}}))\rangle$.

We have

$$(\tau_2\sigma(\theta \cdot E_0))(z) = \tau_2\Big(\frac{1}{4}\sum_{\nu \bmod 4} \theta\big(\frac{z+\nu}{4}\big)E_0\big(\frac{z+\nu}{4}, \frac{s+1}{2}\big)\Big)$$
$$= \tau_2\Big(\frac{1}{4}\sum_{\nu \bmod 4} \theta\big(\frac{z+\nu}{4}\big)E_\infty\big(\tau_2\big(\frac{z+\nu}{4}\big), \frac{s+1}{2}\big)\Big).$$

We have

$$\theta\big(\frac{z+\nu}{4}\big) = \big(\frac{v}{4}\big)^{1/4}\theta_1\big(\frac{z+\nu}{4}\big) = \big(\frac{v}{4}\big)^{1/4}(\theta_1(z) + e^{i\pi\nu/2}\theta_2(z))$$
$$= \frac{1}{\sqrt{2}}(\theta(z) + e^{i\pi\nu/2}\theta_{1/2}(z))$$

where the notations are as in §2.

Using the transformation formulae (2.4) and (2.5) we obtain

$$\begin{aligned}
&(\tau_2\sigma(\theta \cdot E_0))(z)\\
=&\tau_2\big(\frac{1}{4}\sum_{\nu \bmod 4} (\theta(z) + e^{i\pi\nu/2}\theta_{1/2}(z))E_\infty\big(\tau_2\big(\frac{z+\nu}{4}\big), \frac{s+1}{2}\big)\big)\\
=&\frac{1}{4}\sum_{\nu \bmod 4} ((\tau_2\theta)(z) + e^{i\pi\nu/2}(\tau_2\theta_{1/2})(z))E_\infty\Big(\tau_2\big(\frac{\tau_2 z+\nu}{4}\big), \frac{s+1}{2}\Big)\\
=&\frac{1}{4}\sum_{\nu \bmod 4} v^{1/4}((\theta_1(4z) + \theta_2(4z)) + e^{i\pi\nu/2}(\theta_1(4z) - \theta_2(4z)))\\
&\cdot E_\infty\big(\tau_2\big(\frac{\tau_2 z+\nu}{4}\big), \frac{s+1}{2}\big)\\
=&\frac{1}{4}\sum_{\nu \bmod 4} v^{1/4}((1 + e^{i\pi\nu/2})\theta_1(4z) + (1 - e^{i\pi\nu/2})\theta_2(4z))\\
&\cdot E_\infty\big(\tau_2\big(\frac{\tau_2 z+\nu}{4}\big), \frac{s+1}{2}\big).
\end{aligned}$$

Similarly,

$$\begin{aligned}
(\tau_2\sigma(\theta \cdot E_{\frac{1}{2}}))(z) =&\frac{1}{4}\sum_{\nu \bmod 4} v^{1/4}((1 + e^{i\pi\nu/2})\theta_1(4z) + (1 - e^{i\pi\nu/2})\theta_2(4z))\\
&\cdot E_\infty\big(\tau_3^{-1}\big(\frac{\tau_2 z+\nu}{4}\big), \frac{s+1}{2}\big).
\end{aligned}$$

Let

$$E_0^{(\nu)} = E_\infty\big(\tau_2\big(\frac{\tau_2 z+\nu}{4}\big), \frac{s+1}{2}\big) = E_\infty\big(\frac{2z}{-2\nu z+\frac{1}{2}}, \frac{s+1}{2}\big)$$
$$= \sum_{(4x_1,x_3)=1}\Big(\frac{v}{|4x_1\cdot 2z+(-2\nu z+\frac{1}{2})x_3|^2}\Big)^{\frac{s+1}{2}}$$
$$= 2^{s+1}\sum_{(4x_1,x_3)=1}\Big(\frac{v}{|4(4x_1-\nu x_3)z+x_3|^2}\Big)^{\frac{s+1}{2}}$$

and

$$E_{\frac{1}{2}}^{(\nu)} = E_\infty\big(\tau_3^{-1}\big(\frac{\tau_2 z+\nu}{4}\big), \frac{s+1}{2}\big) = E_\infty\Big(\frac{2z}{(-2\nu+4)z+\frac{1}{2}}, \frac{s+1}{2}\Big)$$
$$= 2^{s+1}\sum_{(4x_1,x_3)=1)}\Big(\frac{v}{|4(4x_1-(\nu-2)x_3)z+x_3|^2}\Big)^{\frac{s+1}{2}}$$

Notice that $E_{\frac{1}{2}}^{(\nu)} = E_0^{(\nu-2)}$, and $\sum_{v \bmod 4} E_0^{(\nu)} = E_\infty(z, \frac{s+1}{2})$. We rewrite

$$\begin{aligned}
&2^{-s-1}(\tau_2\sigma(\theta\cdot E_0)+\tau_2\sigma(\theta\cdot E_{\frac{1}{2}}))\\
&= \frac{1}{4}\sum_{\nu \bmod 4} v^{1/4}((1+e^{i\pi\nu/2})\theta_1(4z)+(1-e^{i\pi\nu/2})\theta_2(4z))E_0^{(\nu)}\\
&\qquad +((1-e^{i\pi\nu/2})\theta_1(4z)+(1+e^{i\pi\nu/2})\theta_2(4z))E_{\frac{1}{2}}^{(\nu)}\\
&= \frac{1}{4}\sum_{\nu \bmod 4} v^{1/4}2(\theta_1(4z)+\theta_2(4z))E_0^{(\nu)}\\
&= \frac{1}{2}\theta(z)\sum_{v \bmod 4} E_0^{(\nu)} = \frac{1}{2}\theta(z)E_\infty\big(z, \frac{s+1}{2}\big).
\end{aligned}$$

Thus $I_2+I_3 = \frac{1}{2}I_1$, and we have established the following relation:

$$\begin{aligned}
\Omega(s) &= 2^s\pi^{-\frac{s+1}{2}}\Gamma\big(\frac{s+1}{2}\big)\zeta(s+1)(I_1+I_2+I_3)\\
&= 2^s\pi^{-\frac{s+1}{2}}\Gamma\big(\frac{s+1}{2}\big)\zeta(s+1)\frac{3}{2}I_1.
\end{aligned}$$

Now we compute

$$\begin{aligned}
I_1 &= \int_{\Gamma_0(4)\backslash\mathfrak{H}} F(z)\overline{\theta_1(z)}\sum_{\sigma\in\Gamma_\infty\backslash\Gamma_0(4)}\mathrm{Im}(\sigma z)^{\frac{s+1}{2}}\frac{dudv}{v^2}\\
&= \int_{\Gamma_\infty\backslash\mathfrak{H}} v^{1/4}F(z)\overline{\theta_1(z)}v^{\frac{s+1}{2}}\frac{dudv}{v^2}.
\end{aligned}$$

Since $F(z)$ is an eigenfunction of $\Delta_{1/2}$ with the eigenvalue $-\frac{1}{4}-r^2$ we can use the Fourier expansion (0.10) to obtain

$$\begin{aligned}
&\pi^{-\frac{s+1}{2}}2^s\Gamma(\frac{s+1}{2})I_1\\
&=\pi^{-\frac{s+1}{2}}2^{s+1}\Gamma(\frac{s+1}{2})\int_0^\infty\Big(\sum_{n=1}^\infty\rho(n^2)W_{1/4,ir}(4\pi n^2v)\exp(-2\pi n^2v)\Big)v^{\frac{s}{2}-\frac{1}{4}}\frac{dv}{v}\\
&=\pi^{-\frac{s+1}{2}}2^{s+1}\Gamma(\frac{s+1}{2})\Big(\sum_{n=1}^\infty\frac{\rho(n^2)}{n^{s-1/2}}\Big)\int_0^\infty W_{1/4,ir}(4\pi v)\exp(-2\pi v)v^{\frac{s}{2}-\frac{1}{4}}\frac{dv}{v}.
\end{aligned}$$

Using a table integral ([GR] p. 860) we obtain

$$\begin{aligned}
&\pi^{-\frac{s+1}{2}}2^s\Gamma(\frac{s+1}{2})I_1\\
&=\pi^{-\frac{s+1}{2}}2^{s+1}\Gamma(\frac{s+1}{2})(4\pi)^{-(\frac{s}{2}-\frac{1}{4})}\frac{\Gamma(\frac{s}{2}+\frac{1}{4}+ir)\Gamma(\frac{s}{2}+\frac{1}{4}-ir)}{\Gamma(\frac{s+1}{2})}(\sum_{n=1}^\infty\frac{\rho(n^2)}{n^{s-1/2}})\\
&=\pi^{-\frac{s+1}{2}}2\sqrt{2}\pi^{-(\frac{s}{2}-\frac{1}{4})}\Gamma(\frac{s}{2}+\frac{1}{4}+ir)\Gamma(\frac{s}{2}+\frac{1}{4}-ir)(\sum_{n=1}^\infty\frac{\rho(n^2)}{n^{s-1/2}}\\
&=\pi^{-s-1/4}2\sqrt{2}\Gamma(\frac{s}{2}+\frac{1}{4}+ir)\Gamma(\frac{s}{2}+\frac{1}{4}-ir)(\sum_{n=1}^\infty\frac{\rho(n^2)}{n^{s-1/2}}\\
&=2\sqrt{2}\pi^{1/4}\pi^{-s-1/2}\Gamma(\frac{s}{2}+\frac{1}{4}+ir)\Gamma(\frac{s}{2}+\frac{1}{4}-ir)\Big(\sum_{n=1}^\infty\frac{\rho(n^2)}{n^{s-1/2}}\Big),
\end{aligned}$$

and hence

$$\begin{aligned}
\Omega(s)=&2^s\pi^{-\frac{s+1}{2}}\Gamma(\frac{s+1}{2})\zeta(s+1)\frac{3}{2}I_1\\
=&\frac{3}{2}\sqrt{2}\pi^{1/4}2\pi^{-s-1/2}\Gamma(\frac{s}{2}+\frac{1}{4}+ir)\\
&\cdot\Gamma(\frac{s}{2}+\frac{1}{4}-ir)\zeta(s+1)\Big(\sum_{n=1}^\infty\frac{\rho(n^2)}{n^{s-1/2}}\Big).
\end{aligned}$$

Thus we have established the following formula:

$$\Omega(s)=3\sqrt{2}\pi^{\frac{1}{4}}\big(\pi^{-s-1/2}\Gamma(\frac{s}{2}+\frac{1}{4}+ir)\Gamma(\frac{s}{2}+\frac{1}{4}-ir)\sum_{n=1}^\infty\frac{b(n)}{n^s}\big),\tag{4.1}$$

where

$$b(k)=\sum_{nm=k}\rho(m^2)n^{-1}m^{1/2}.\tag{4.2}$$

The expression in (4.1) is recognized (see Proposition 1.3) to be the Mellin transform of an even Maass cusp form of weight 0 with Fourier expansion $6\sqrt{2}\pi^{\frac{1}{4}}\sum_{n=1}^{\infty} b(n)W_{0,2ir}(4\pi ny)\cos(2\pi nx)$, so $\psi \in U$, i.e. part (i) is proved, and ψ is an eigenfunction of Δ_0 with eigenvalue $-\frac{1}{4}-(2r)^2$. We now use (4.2) to show that ψ is an eigenform of T_p for $p \neq 2$. Once this is established, parts (ii) and (iii) follow from Proposition 1.2 together with (4.1).

To do so we need to show that a function with Fourier expansion $\psi_1 = 2\sum_{n=1}^{\infty} b(n)W_{0,2ir}(4\pi ny)\cos(2\pi nx)$ is a common eigenfunction of all Hecke operators T_p providing $F(z)$ is a common eigenfunction of all Hecke operators T_{p^2} for $p \neq 2$. Let $p \neq 2$. Denoting the corresponding eigenvalues by $\mu(p)$ and using (1.3), we have the following relation between the Fourier coefficients of $F(z)$:

$$\mu(p)\rho(n) = p\rho(np^2) + p^{-1/2}\left(\frac{n}{p}\right)\rho(n) + p^{-1}\rho\left(\frac{n}{p^2}\right), \tag{4.3}$$

and we want to show that the following relation holds for the Fourier coefficients $a(k)$ of ψ_1:

$$\mu(p)b(k) = p^{1/2}b(kp) + p^{-1/2}b\left(\frac{k}{p}\right). \tag{4.4}$$

First, let $(k,p)=1$. Then from (4.3) we get

$$\rho(p^2n) = p^{-1}\mu(p)\rho(n) - p^{-3/2}\left(\frac{n}{p}\right)\rho(n) - p^{-2}\rho\left(\frac{n}{p^2}\right),$$

and hence

$$\begin{aligned} b(kp) &= \sum_{nm=kp} \rho(m^2)n^{-1}m^{1/2} \\ &= \sum_{n(mp)=kp} \rho(m^2p^2)n^{-1}m^{1/2}p^{1/2} + \sum_{(np)m=kp} \rho(m^2)n^{-1}p^{-1}m^{1/2} \\ &= \sum_{nm=k} \rho(m^2)(p^{-1/2}\mu(p) - p^{-1} + p^{-1})n^{-1}m^{1/2} = p^{-1/2}\mu(p)b(k), \end{aligned}$$

and since in this case $b(\frac{k}{p}) = 0$ we obtain the required identity (4.4). Now let $e \neq 0$ be the maximal power such that $p^e | k$. Then

$$b(k) = \sum_{nm=\frac{k}{p^e}} \sum_{j=0}^{e} \rho(m^2p^{2j})n^{-1}p^{-e+j}m^{1/2}p^{j/2}.$$

Let $A_j = \rho(m^2p^{2j})$. From (4.3) we have

$$\rho(p^2n) = p^{-1}\mu(p)\rho(n) - p^{-3/2}\Big(\frac{n}{p}\Big)\rho(n) - p^{-2}\rho\Big(\frac{n}{p^2}\Big),$$

and hence we have

$$A_1 = \rho(m^2p^2) = p^{-1}\mu(p)\rho(m^2) - p^{-3/2}\rho(m^2) = p^{-1}(\mu(p) - p^{-1/2})\rho(m^2).$$

Notice that for $j \geq 2$,

$$\Big(\frac{m^2p^{2j-2}}{p}\Big) = 0,$$

and hence

$$A_j = \rho(m^2p^{2j}) = \rho(p^2m^2p^{2j-2}) = p^{-1}\mu(p)\rho(m^2p^{2j-2}) - p^{-2}\rho(m^2p^{2j-4}),$$

i.e. we have the following recurrence formula:

$$A_j = p^{-1}(\mu(p)A_{j-1} - p^{-1}A_{j-2}).$$

It is easily proved by induction that

$$A_j = p^{-j}P_j(\mu(p))A_0 = p^{-j}P_j(\mu(p))\rho(m^2),$$

and P_j are polynomials in $\mu(p)$ with coefficients depending on p^{-1}:

$$P_0 = 1, \quad P_1(x) = x - p^{-1}, \quad P_2(x) = x^2 - xp^{-1} - 1, \ \dots \tag{4.5}$$

satisfying the following recurrence formula:

$$P_{j+1}(x) + P_{j-1}(x) = xP_j(x). \tag{4.6}$$

We can rewrite $b(k)$ as follows:

$$\begin{aligned} b(k) &= \sum_{mn=\frac{k}{p^e}} \sum_{j=0}^{e} p^{-j}P_j(\mu(p))\rho(m^2)n^{-1}p^{-e+j}m^{1/2}p^{j/2} \\ &= \Big(\sum_{nm=\frac{k}{p^e}} \rho(m^2)n^{-1}m^{1/2}\Big)p^{-e}\sum_{j=0}^{e} P_j(\mu(p))p^{j/2} \\ &= b\Big(\frac{k}{p^e}\Big)p^{-e}\sum_{j=0}^{e} P_j(\mu(p))p^{j/2}. \end{aligned}$$

Similarly,

$$b\Big(\frac{k}{p}\Big) = b\Big(\frac{k}{p^e}\Big)p^{-e+1}\sum_{j=0}^{e-1} P_j(\mu(p))p^{j/2},$$

$$b(kp) = b\Big(\frac{k}{p^e}\Big)p^{-e-1}\sum_{j=0}^{e+1} P_j(\mu(p))p^{j/2}.$$

$$\mu(p)b(k) = b\Big(\frac{k}{p^e}\Big)p^{-e}\sum_{j=0}^{e} \mu(p)P_j(\mu(p))p^{j/2}$$

$$(4.7)\qquad = b\Big(\frac{k}{p^e}\Big)p^{-e}\Big(\sum_{j=1}^{e} \mu(p)P_j(\mu(p))p^{j/2} + \mu(p)P_0(\mu(p))\Big).$$

$$p^{1/2}b(kp) = b\Big(\frac{k}{p^e}\Big)p^{-e}\sum_{j=0}^{e+1} P_j(\mu(p))p^{(j-1)/2} = b\Big(\frac{k}{p^e}\Big)p^{-e}\sum_{i=-1}^{e} P_{i+1}(\mu(p))p^{i/2}$$

$$(4.8)\qquad = b\Big(\frac{k}{p^e}\Big)p^{-e}\Big(\sum_{i=1}^{e} P_{i+1}(\mu(p))p^{i/2} + P_0(\mu(p))p^{-1/2} + P_1(\mu(p))\Big).$$

$$(4.9)\quad p^{-1/2}b\Big(\frac{k}{p}\Big) = b\Big(\frac{k}{p^e}\Big)p^{-e}\sum_{j=0}^{e-1} P_j(\mu(p))p^{(j+1)/2} = b\Big(\frac{k}{p^e}\Big)p^{-e}\sum_{i=1}^{e} P_{i-1}(\mu(p))p^{i/2}.$$

We want to check that the expression (4.7) is equal to the sum of the expressions (4.8) and (4.9). By the formula (4.6) for $j = 1,\dots,e$, $\mu(p)P_j(\mu(p)) = P_{j+1}(\mu(p)) + P_{j-1}(\mu(p))$, hence the sums in the corresponding expressions are equal. Also $P_0(\mu(p))p^{-1/2} + P_1(\mu(p)) = \mu(p)P_0(\mu(p))$ by (4.5), and this completes the proof of the formula (4.4) for $p \neq 2$. Thus we proved that ψ is a common eigenfunction of T_p, $p \neq 2$, and Δ_0.

As follows from the Strong Multiplicity 1 Theorem for $SL_2(\mathbb{Z})$ (Proposition 1.2), ψ is also an eigenfunction of T_2; if $\rho(1) = 0$, $\psi \equiv 0$, if $\rho(1) \neq 0$ it is a multiple of a normalized Maass cusp form with Fourier expansion

$$2\sum_{n=1}^{\infty} a(n)W_{0,2ir}(4\pi ny)\cos(2\pi nx)$$

whose Fourier coefficients are defined from the following equation:

$$\zeta(s+1)\sum_{n=1}^{\infty}\frac{\rho(n^2)}{n^{s-1/2}} = \rho(1)\sum_{n=1}^{\infty} a(n)n^{-s}.$$

With this the proof of Proposition 4.1 is complete. ∎

5. Proof of the Theorem

Let $\varphi \in U$, φ an even Maass–Hecke normalized cusp form, and

$$F(z) = \int_{\Gamma\backslash G} \varphi(g)\Theta(z,g)dg \tag{5.1}$$

(notice that if φ is odd then $F \equiv 0$ since Θ is even by Proposition 2.2 (iii)). Then by Proposition 2.3, $F \in V^+$, and it is an eigenfunction of $\Delta_{1/2}$.

Now let $\{F_j\}$, $j = 1, 2, \ldots$ be an orthonormal basis for V^+ consisting of eigenfunctions of T_{p^2} and $\Delta_{1/2}$ and also of $L = \tau_2\sigma$ with eigenvalue 1. We may expand

$$F(z) = \sum_{j=1}^{\infty} \langle F, F_j\rangle_{\Gamma_0(4)} F_j(z). \tag{5.2}$$

Of course, since F has a fixed eigenvalue for $\Delta_{1/2}$, the sum in (5.2) is, in fact, finite. Now

$$\begin{aligned} \langle F, F_j\rangle &= \int_{\Gamma_0(4)\backslash\mathfrak{H}} F(z)\overline{F_j(z)}\frac{dudv}{v^2} \\ &= \int_{\Gamma_0(4)\backslash\mathfrak{H}} \Big(\int_{\Gamma\backslash G} \varphi(g)\Theta(z,g)dg\Big)\overline{F_j(z)}\frac{dudv}{v^2} \\ &= \int_{\Gamma\backslash\mathfrak{H}} \varphi(g)\overline{\Big(\int_{\Gamma_0(4)\backslash\mathfrak{H}} F_j(z)\overline{\Theta(z,g)}\frac{dudv}{v^2}\Big)}dg. \end{aligned} \tag{5.3}$$

According to Proposition 4.1 the inner integral gives

$$3\sqrt{2}\pi^{1/4}\rho_j(1)\psi_j(g) \tag{5.4}$$

where $\psi_j(g)$ is a Hecke normalized Maass cusp form in U. Thus

$$\begin{aligned} \langle F, F_j\rangle &= 3\sqrt{2}\pi^{1/4}\overline{\rho_j(1)}\langle\varphi, \psi_j\rangle_\Gamma \\ &= \begin{cases} 3\sqrt{2}\pi^{1/4}\overline{\rho_j(1)}\langle\varphi,\varphi\rangle & \text{if } \mathrm{Shim}(F_j) = \varphi, \\ 0 & \text{otherwise,} \end{cases} \end{aligned} \tag{5.5}$$

i.e. we have

$$F(z) = \sum_{\mathrm{Shim}(F_j)=\varphi} 3\sqrt{2}\pi^{1/4}\overline{\rho_j(1)}\langle\varphi,\varphi\rangle F_j(z). \tag{5.6}$$

Now compare the nth Fourier coefficient on both sides. We get

$$\begin{aligned}(5.7)\qquad &\int_0^1 \Big(\int_{\Gamma\backslash G} \varphi(g)\Theta(z,g)dg\Big)e(-nu)du \\ &= 3\sqrt{2}\pi^{1/4}\langle\varphi,\varphi\rangle \sum_{\mathrm{Shim}(F_j)=\varphi} \overline{\rho_j(1)}\rho_j(n)W_{1/4\mathrm{sgn}(n),ir}(4\pi|n|v).\end{aligned}$$

Let $v \to \infty$. By geometric calculation (3.13), the left hand side of (5.7) is asymptotic to

$$\frac{v^{-1/4}}{8\pi|n|}e^{-2\pi v|n|}\Big(\sum_{z\in\Lambda_n}{}^{*}\varphi(z)\Big), \quad \text{if } n<0.$$

The right hand side is asymptotic to

$$3\sqrt{2}\pi^{1/4}\langle\varphi,\varphi\rangle\Big(\sum_{\mathrm{Shim}(F_j)=\varphi} \overline{\rho_j(1)}\rho_j(n)\Big)(4\pi|n|v)^{-1/4}e^{-2\pi|n|v}.$$

Hence for $n<0$ we get

$$\begin{aligned}\frac{1}{\langle\varphi,\varphi\rangle}\sum_{z\in\Lambda_n}{}^{*}\varphi(z) &= 3\sqrt{2}\pi^{1/4}|n|^{3/4}\pi^{3/4}4^{5/4} \sum_{\mathrm{Shim}(F_j)=\varphi} \overline{\rho_j(1)}\rho_j(n) \\ &= 24\pi|n|^{3/4} \sum_{\mathrm{Shim}(F_j)=\varphi} \overline{\rho_j(1)}\rho_j(n).\end{aligned}$$

Similarly, for $n>0$, as $v\to\infty$, by geometric calculation (3.28) the left hand side of (5.7) is asymptotic to

$$\frac{v^{1/4}e^{-2\pi vn}}{2\sqrt{n}}\Big(\sum_{C\in\Lambda_n}\int_C \varphi ds\Big)$$

while the right hand side of (5.7) is asymptotic to

$$3\sqrt{2}\pi^{1/4}\langle\varphi,\varphi\rangle\Big(\sum_{\mathrm{Shim}(F_j)=\varphi} \overline{\rho_j(1)}\rho_j(n)\Big)(4\pi nv)^{1/4}e^{-2\pi nv}.$$

Hence for $n>0$ we get

$$\begin{aligned}\frac{1}{\langle\varphi,\varphi\rangle}\sum_{C\in\Lambda_n}\int_C \varphi ds &= 3\sqrt{2}\pi^{1/4}n^{3/4}\pi^{1/4}2^{3/2} \sum_{\mathrm{Shim}(F_j)=\varphi} \overline{\rho_j(1)}\rho_j(n) \\ &= 12\pi^{1/2}n^{3/4} \sum_{\mathrm{Shim}(F_j)=\varphi} \overline{\rho_j(1)}\rho_j(n). \qquad \blacksquare\end{aligned}$$

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